One-dimensional Heisenberg XXZ magnetic chain with isolated impurities

1990 J. Phys. A: Math. Gen. 23 2107

(http://iopscience.iop.org/0305-4470/23/11/031)

J. Phys. A: Math. Gen. 23 (1990) 2107-2113. Printed in the UK

One-dimensional Heisenberg *XXZ* magnetic chain with isolated impurities

Lin-Jie Jiang and Huan-Qiang Zhou

Centre of Theoretical Physics, CCAST (World Laboratory), Beijing 100080, People's Republic of China, and Institute of Solid State Physics, Sichuan Normal University, Chengdu 610066, People's Republic of China

Received 6 September 1989, in final form 25 October 1989

Abstract. The one-dimensional (1D) Heisenberg *XXZ* magnetic chain with isolated impurities is presented as a new completely integrable system. This system includes the 1D Heisenberg *XXZ* model and the 1D Heisenberg *XXX* magnetic chain with isolated impurities as special limiting cases. The model Hamiltonian is diagonalised, and the algebraic Bethe ansatz equations are derived. This makes it possible to investigate the equilibrium thermodynamics of the system.

1. Introduction

In the last decade, considerable progress has been made in the study of the theory of quantum completely integrable systems. At present, a number of completely integrable systems in (1+1)-dimensional quantum field theories and in two-dimensional (2D) lattice statistical mechanics are known which are soluble by means of the Bethe ansatz method (BA) (Bethe 1931, Lieb and Liniger 1963, Yang 1967, Belavin 1979, Wiegmann 1980, Andrei *et al* 1983 and references therein) or the quantum inverse scattering method (QISM) (Thacker 1981, Izergin and Korepin 1982, Kulish and Sklyanin 1982). Among these systems, lattice spin models have received much attention for their relevance to the low-dimensional condensed matter theory. On the other hand, they have often served as prototypes for the study of other quantum integrable systems. A recent remarkable example is the demonstration of integrability of the principal field model in two dimensions (Faddeev and Reshetikhin 1985).

In this paper we present a new completely integrable system describing the interaction of the usual Heisenberg *XXZ* magnetic chain with isolated impurities. Our model includes some physically interesting models as special limiting cases, such as the 1D Heisenberg *XXZ* model and the 1D isotropic magnetic chain with impurities recently proposed by Andrei and Johannesson (1984). In the latter case, the Lax pair has been constructed by us in a recent work (Zhou and Jiang 1989), which provides a direct demonstration for the integrability of the system.

2. Formulation of the model

Let us consider the interaction of the Heisenberg *XXZ* magnetic chain with an impurity located on the mth link (see figure 1), with dynamics determined by the Hamiltonian

$$H = H_0 + H_{(m,S)} \tag{1}$$

0305-4470/90/112107+07$03.50

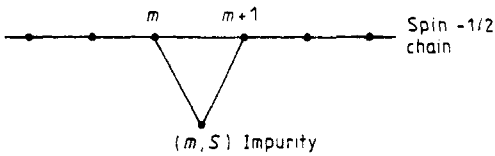

Figure 1. Pictorial representation of an isolated impurity interacting with the Heisenberg XXZ magnetic chain.

where

$$H_0 = J \sum_{j=1}^{N} \left(\sigma_{j+1}^{+}\sigma_j^{-} + \sigma_{j+1}^{-}\sigma_j^{+} + \frac{\cosh\gamma}{2}\sigma_{j+1}^{z}\sigma_j^{z} \right) \tag{2}$$

and

$$\begin{aligned} H_{(m,s)} = \frac{J\sinh^2\gamma}{\sinh^2[(l+1)\gamma/2]} \Bigg[& \cosh[\gamma(1+S^z)/2](\sigma_{m+1}^{+}+\sigma_m^{+})S^{-} \\ & +(\sigma_{m+1}^{-}+\sigma_m^{-})S^{+}\cosh[\gamma(1+S^z)/2] \\ & +\frac{\cosh\gamma}{2\sinh\gamma}(\sigma_{m+1}^{z}+\sigma_m^{z})\sinh(\gamma S^z)+\sigma_{m+1}^{+}\sigma_m^{+}S^{-}S^{-}+\sigma_{m+1}^{-}\sigma_m^{-}S^{+}S^{+} \\ & +\frac{\cosh\gamma}{\sinh\gamma}[\sinh[\gamma(1+S^z)/2](\sigma_{m+1}^{+}\sigma_m^{z}+\sigma_{m+1}^{z}\sigma_m^{+})S^{-} \\ & +(\sigma_{m+1}^{-}\sigma_m^{z}+\sigma_{m+1}^{z}\sigma_m^{-})S^{+}\sinh[\gamma(1+S^z)/2]] \\ & +\frac{1}{\sinh^2\gamma}[\sinh[\gamma(1+S^z)/2]\sinh[\gamma(1-S^z)/2]-\sinh^2[(l+1)\gamma/2]] \\ & \times(\sigma_{m+1}^{+}\sigma_m^{-}+\sigma_{m+1}^{-}\sigma_m^{+})-\frac{1}{4\sinh^2\gamma}[\sinh^2\gamma(S^{+}S^{-}+S^{-}S^{+}) \\ & -\cosh\gamma\cosh(\gamma S^z)+2\cosh\gamma\sinh^2[(l+1)\gamma/2]+1]\sigma_{m+1}^{z}\sigma_m^{z} \\ & -\frac{1}{4}\left(\cosh\gamma+\frac{2\sinh\gamma\cosh[(l+1)\gamma/2]}{\sinh[(l+1)\gamma/2]}\right)(S^{+}S^{-}+S^{-}S^{+}) \\ & -\frac{2\cosh[(l-1)\gamma/2]+\cosh^2\gamma\sinh[(l+1)\gamma/2]}{4\sinh\gamma\sinh[(l+1)\gamma/2]}\cosh(\gamma S^z)\Bigg]. \end{aligned} \tag{3}$$

Here the periodic boundary conditions are assumed. $\sigma_j^{\pm} = \frac{1}{2}(\sigma_j^x \pm \mathrm{i}\sigma_j^y)$ and σ_j^x, σ_j^y, σ_j^z are the usual Pauli spin operators at lattice site j, while the operators S^+, S^- and S^z assigned to the impurity satisfy the following commutation relations:

$$[S^z, S^{\pm}] = \pm 2S^{\pm} \qquad [S^+, S^-] = \frac{\sinh(\gamma S^z)}{\sinh\gamma}. \tag{4}$$

Obviously, our model includes the 1D XXZ model and the 1D XXX magnetic chain with a spin-$l/2$ impurity as special cases. Note that by assigning spin-$\frac{1}{2}$ to the impurity site, the Hamiltonian (1) reduces properly to that of the Heisenberg *XXZ* model for a spin-$\frac{1}{2}$ chain of $N+1$ sites. Also note that by taking a special limit ($\gamma \to 0$), the Hamiltonian (1) reduces to the form

$$H = H_0 + H_{(m,S)} \tag{5}$$

with

$$H_0 = J \sum_{j=1}^{N} (\sigma_{j+1}^{+}\sigma_j^{-} + \sigma_{j+1}^{-}\sigma_j^{+} + \tfrac{1}{2}\sigma_{j+1}^{z}\sigma_j^{z}) \tag{6}$$

and

$$\begin{aligned} H_{(m,S)} = \frac{4J}{(l+1)^2} \Big\{ & (\sigma_{m+1}^{+} + \sigma_m^{+})S^{-} + (\sigma_{m+1}^{-} + \sigma_m^{-})S^{+} + \tfrac{1}{2}(\sigma_{m+1}^{z} + \sigma_m^{z})S^{z} + \sigma_{m+1}^{+}\sigma_m^{+}S^{-}S^{-} \\ & + \sigma_{m+1}^{-}\sigma_m^{-}S^{+}S^{+} + (\sigma_{m+1}^{+}\sigma_m^{z} + \sigma_{m+1}^{z}\sigma_m^{+})\frac{1+S^z}{2}S^{-} \\ & + (\sigma_{m+1}^{-}\sigma_m^{z} + \sigma_{m+1}^{z}\sigma_m^{-})S^{+}\frac{1+S^z}{2} - \tfrac{1}{4}[l(l+2) + (S^z)^2](\sigma_{m+1}^{+}\sigma_m^{-} + \sigma_{m+1}^{-}\sigma_m^{+}) \\ & - \tfrac{1}{4}[l(l+2) - (S^z)^2]\sigma_{m+1}^{z}\sigma_m^{z} \Big\} \end{aligned} \tag{7}$$

which is nothing but the Hamiltonian describing the interaction of a spin-$l/2$ impurity with an isotropic Heisenberg chain (Andrei and Johannesson 1984).

The coupling constants between sites $m, (m, S)$ and $m+1$ in $H_{(m,S)}$ have been chosen to assure the integrability of the model. Indeed, the Hamiltonian (1) may be related to a class of commuting transfer matrices $\tau(\lambda)$ via the so-called Baxter–Lüscher relation:

$$H = J \sinh \gamma \frac{\partial}{\partial \lambda} \ln \tau(\lambda) \Big|_{\lambda=0}. \tag{8}$$

In our case, $\tau(\lambda)$ may be constructed as follows

$$\tau(\lambda) = \mathrm{tr}[L_N^{(1/2)}(\lambda) \dots L_{m+1}^{(1/2)}(\lambda) L_{(m,S)}^{(l/2)}(\lambda) L_m^{(1/2)}(\lambda) \dots L_1^{(1/2)}(\lambda)] \tag{9}$$

with

$$L_j^{(1/2)}(\lambda) = \frac{1}{\sinh(\lambda+\gamma)} \begin{pmatrix} \sinh\left(\lambda + \gamma \frac{1+\sigma_j^z}{2}\right) & \sinh \gamma\sigma_j^{-} \\ \sinh \gamma\sigma_j^{+} & \sinh\left(\lambda + \gamma \frac{1-\sigma_j^z}{2}\right) \end{pmatrix} \tag{10}$$

and

$$L_{(m,S)}^{(l/2)}(\lambda) = \frac{1}{\sinh(\lambda + \frac{1}{2}(l+1)\gamma)} \begin{pmatrix} \sinh\left(\lambda + \gamma \frac{1+S^z}{2}\right) & \sinh \gamma S^{-} \\ \sinh \gamma S^{+} & \sinh\left(\lambda + \gamma \frac{1-S^z}{2}\right) \end{pmatrix}. \tag{11}$$

It must be noticed that $L_j^{(1/2)}(\lambda)$ and $L_{(m,S)}^{(l/2)}(\lambda)$ satisfy the Yang-Baxter relations (Faddeev and Reshetikhin 1985, Kirillov and Reshetikhin 1987)

$$\begin{aligned} R(\lambda-\mu)L_j^{(1/2)}(\lambda)\otimes L_j^{(1/2)}(\mu) &= L_j^{(1/2)}(\mu)\otimes L_j^{(1/2)}(\lambda)R(\lambda-\mu) \\ R(\lambda-\mu)L_{(m,S)}^{(l/2)}(\lambda)\otimes L_{(m,S)}^{(l/2)}(\mu) &= L_{(m,S)}^{(l/2)}(\mu)\otimes L_{(m,S)}^{(l/2)}(\lambda)R(\lambda-\mu) \end{aligned} \tag{12}$$

with

$$R(\lambda-\mu)=\begin{pmatrix} \sinh(\lambda-\mu+\gamma) & 0 & 0 & 0 \\ 0 & \sinh\gamma & \sinh(\lambda-\mu) & 0 \\ 0 & \sinh(\lambda-\mu) & \sinh\gamma & 0 \\ 0 & 0 & 0 & \sinh(\lambda-\mu+\gamma) \end{pmatrix}. \tag{13}$$

Here by $\otimes$ we mean the matrix direct product,

$$(A\otimes B)_{ik,jl}=A_{ij}B_{kl}$$

whence the commutativity of the transfer matrices for different values of the spectral parameter λ immediately follows (Kulish and Sklyanin 1982, Andrei and Johannesson 1984).

To close this section we give some formulae about the operators S^+, S^-, and S^z which have been used in the derivation of (8):

$$S^\pm\, e^{\alpha S^z}=e^{\mp 2\alpha}\, e^{\alpha S^z} S^\pm \tag{14}$$

$$\begin{aligned} S^\pm\sinh(\alpha+\beta S^z) &= \sinh(\alpha\mp 2\beta+\beta S^z)S^\pm \\ S^\pm\cosh(\alpha+\beta S^z) &= \cosh(\alpha\mp 2\beta+\beta S^z)S^\pm \end{aligned} \tag{15}$$

$$\begin{aligned} \sinh(\alpha+\beta S^z)S^\pm &= S^\pm\sinh(\alpha\pm 2\beta+\beta S^z) \\ \cosh(\alpha+\beta S^z)S^\pm &= S^\pm\cosh(\alpha\pm 2\beta+\beta S^z) \end{aligned} \tag{16}$$

$$\cosh\frac{\gamma(1\mp S^z)}{2}S^\pm = S^\pm\cosh\frac{\gamma(1\pm S^z)}{2} \tag{17}$$

$$\sinh\frac{\gamma(1\mp S^z)}{2}S^\pm = -S^\pm\sinh\frac{\gamma(1\pm S^z)}{2}. \tag{18}$$

3. Algebraic Bethe ansatz equations

Let us now turn our attention to the diagonalisation of the model Hamiltonian (1). This can be done by using the algebraic Bethe ansatz method initially formulated by Faddeev, Korepin, Sklyanin and their collaborators (Faddeev 1980, Izergin and Korepin 1982, Kulish and Sklyanin 1982, de Vega and Lopes 1987, Zhou *et al* 1989). For our purpose, it is convenient to introduce a monodromy matrix

$$T(\lambda)=L_N^{(1/2)}(\lambda)\ldots L_{m+1}^{(1/2)}(\lambda)L_{(m,S)}^{(l/2)}(\lambda)L_m^{(1/2)}(\lambda)\ldots L_1^{(1/2)}(\lambda)$$

whose trace is the transfer matrix $\tau(\lambda)=\operatorname{tr}T(\lambda)$. From (12), it follows that

$$R(\lambda-\mu)T(\lambda)\otimes T(\mu)=T(\mu)\otimes T(\lambda)R(\lambda-\mu). \tag{19}$$

Hence, representing the monodromy matrix as

$$T(\lambda)=\begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{20}$$

and equating the corresponding elements on both sides of (19), we have

$$[A(\lambda), A(\mu)] = [B(\lambda), B(\mu)] = [C(\lambda), C(\mu)] = [D(\lambda), D(\mu)] = 0$$

$$A(\lambda)B(\mu) = \frac{\sinh(\lambda-\mu-\gamma)}{\sinh(\lambda-\mu)} B(\mu)A(\lambda) + \frac{\sinh\gamma}{\sinh(\lambda-\mu)} B(\lambda)A(\mu) \tag{21}$$

$$D(\lambda)B(\mu) = \frac{\sinh(\lambda-\mu+\gamma)}{\sinh(\lambda-\mu)} B(\mu)D(\lambda) - \frac{\sinh\gamma}{\sinh(\lambda-\mu)} B(\lambda)D(\mu).$$

These commutation relations provide a basis for finding the exact eigenvalues and eigenvectors of the Hamiltonian (1). As usual we introduce a pseudovacuum state defined by

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}_N \otimes \ldots \otimes \begin{pmatrix}1\\0\end{pmatrix}_{m+1} \otimes \begin{pmatrix}1\\0\\ \vdots \\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix}_m \otimes \ldots \otimes \begin{pmatrix}1\\0\end{pmatrix}_1. \tag{22}$$

Using $T(\lambda)$ on this state, we have

$$A(\lambda)|0\rangle = |0\rangle$$

$$D(\lambda)|0\rangle = \left(\frac{\sinh\gamma}{\sinh(\lambda+\gamma)}\right)^N \frac{\sinh(\lambda+\frac{1}{2}(1-l)\gamma)}{\sinh(\lambda+\frac{1}{2}(1+l)\gamma)}|0\rangle \tag{23}$$

$$C(\lambda)|0\rangle = 0.$$

Evidently, we may consider $B(\lambda)$ and $C(\lambda)$ as creation and annihilation operators respectively for elementary excitations over the pseudovacuum $|0\rangle$. Thus, one may seek for eigenstates of the transfer matrix $\tau(\lambda)$ of the form

$$|\Omega\rangle = \prod_{i=1}^{M} B(\lambda_i)|0\rangle. \tag{24}$$

From the commutation relations (21), it follows that

$$A(\lambda)|\Omega\rangle = \Lambda_+(\lambda; \lambda_1, \ldots, \lambda_M)|\Omega\rangle + \sum_{i=1}^{M} \Lambda_i(\lambda; \lambda_1, \ldots, \lambda_M)|\Omega^{(i)}\rangle \tag{25}$$

$$D(\lambda)|\Omega\rangle = \Lambda_-(\lambda; \lambda_1, \ldots, \lambda_M)|\Omega\rangle + \sum_{i=1}^{M} \Lambda_i'(\lambda; \lambda_1, \ldots, \lambda_M)|\Omega^{(i)}\rangle \tag{26}$$

where

$$|\Omega^{(i)}\rangle = \prod_{j\neq i} B(\lambda_j)B(\lambda)|0\rangle \tag{27}$$

$$\Lambda_+(\lambda; \lambda_1, \ldots, \lambda_M) = \prod_{i=1}^{M} \frac{\sinh(\lambda-\lambda_i-\gamma)}{\sinh(\lambda-\lambda_i)} \tag{28}$$

$$\Lambda_-(\lambda; \lambda_1, \ldots, \lambda_M) = \left(\frac{\sinh\lambda}{\sinh(\lambda+\gamma)}\right)^N \frac{\sinh(\lambda+\frac{1}{2}(1-l)\gamma)}{\sinh(\lambda+\frac{1}{2}(1+l)\gamma)} \prod_{i=1}^{M} \frac{\sinh(\lambda-\lambda_i+\gamma)}{\sinh(\lambda-\lambda_i)} \tag{29}$$

$$\Lambda_i(\lambda; \lambda_1, \ldots, \lambda_M) = \frac{\sinh\gamma}{\sinh(\lambda-\lambda_i)} \prod_{j\neq i} \frac{\sinh(\lambda_i-\lambda_j-\gamma)}{\sinh(\lambda_i-\lambda_j)} \tag{30}$$

$$\Lambda_i'(\lambda;\lambda_1,\ldots,\lambda_M) = -\left(\frac{\sinh\lambda_i}{\sinh(\lambda_i+\gamma)}\right)^N \frac{\sinh(\lambda_i+\frac{1}{2}(1-l)\gamma)}{\sinh(\lambda_i+\frac{1}{2}(1+l)\gamma)} \frac{\sinh\gamma}{\sinh(\lambda-\lambda_i)} \prod_{j\neq i} \frac{\sinh(\lambda_i-\lambda_j+\gamma)}{\sinh(\lambda_i-\lambda_j)}. \tag{31}$$

Thus, $|\Omega\rangle$ is an eigenvector of $\tau(\lambda)$ with the eigenvalue

$$\Lambda(\lambda;\lambda_1,\ldots,\lambda_M)=\Lambda_+(\lambda;\lambda_1,\ldots,\lambda_M)+\Lambda_-(\lambda;\lambda_1,\ldots,\lambda_M) \tag{32}$$

provided the following equations hold

$$\Lambda_i+\Lambda_i'=0 \qquad i=1,2,\ldots,M$$

or

$$\left(\frac{\sinh\lambda_i}{\sinh(\lambda_i+\gamma)}\right)^N \frac{\sinh(\lambda_i+\frac{1}{2}(1-l)\gamma)}{\sinh(\lambda_i+\frac{1}{2}(1+l)\gamma)} = -\prod_{j=1}^{M} \frac{\sinh(\lambda_i-\lambda_j-\gamma)}{\sinh(\lambda_i-\lambda_j+\gamma)}. \tag{33}$$

These are nothing but the Bethe ansatz equations. From (8) we see that $|\Omega\rangle$ is also an eigenstate of the Hamiltonian (1). Since the transfer matrices commute, the eigenvectors are λ-independent and then the energy eigenvalues can be determined from the logarithmic derivative of $\Lambda(\lambda)$. From (8) and (32) we have

$$E = J\sinh^2\gamma \sum_{i=1}^{M} \frac{1}{\sinh(\lambda_i+\gamma)\sinh\lambda_i}. \tag{34}$$

Finally, we point out that it is convenient to shift the parameters λ_i by $\gamma/2$. Thus, we finally get the algebraic Bethe ansatz equations

$$\left(\frac{\sinh(\lambda_i-\frac{1}{2}\gamma)}{\sinh(\lambda_i+\frac{1}{2}\gamma)}\right)^N \frac{\sinh(\lambda_i-l\gamma/2)}{\sinh(\lambda_i+l\gamma/2)} = -\prod_{j=1}^{M} \frac{\sinh(\lambda_i-\lambda_j-\gamma)}{\sinh(\lambda_i-\lambda_j+\gamma)} \qquad i=1,2,\ldots,M \tag{35}$$

and the energy eigenvalue

$$E = 2J\sinh^2\gamma \sum_{i=1}^{M} \frac{1}{\cosh 2\lambda_i - \cosh\gamma}. \tag{36}$$

4. Conclusion

So far we have shown that the 1D Heisenberg XXZ magnetic chain with an isolated impurity is completely integrable. Obviously, our results can be readily extended to the case of an arbitrary number of isolated impurities. Here, by 'isolated' we mean that the following conditions hold:

$$m_\alpha + 1 \leq m_{\alpha+1} \qquad \alpha = 1,2,\ldots,K-1. \tag{37}$$

Then, the Hamiltonian reads

$$H = H_0 + \sum_{\alpha=1}^{K} H_{(m_\alpha,S_\alpha)} \tag{38}$$

with H_0 and $H_{(m_\alpha,S_\alpha)}$ being defined by (2) and (3), respectively. The corresponding Bethe ansatz equations are

$$\left(\frac{\sinh(\lambda_i-\frac{1}{2}\gamma)}{\sinh(\lambda_i+\frac{1}{2}\gamma)}\right)^N \prod_{\alpha=1}^{K} \frac{\sinh(\lambda_i-\frac{1}{2}l_\alpha\gamma)}{\sinh(\lambda_i+\frac{1}{2}l_\alpha\gamma)} = -\prod_{j=1}^{M} \frac{\sinh(\lambda_i-\lambda_j-\gamma)}{\sinh(\lambda_i-\lambda_j+\gamma)} \qquad i=1,2,\ldots,M \tag{39}$$

with the energy eigenvalue

$$E = 2J \sinh^2 \gamma \sum_{i=1}^{M} \frac{1}{\cosh 2\lambda_i - \cosh \gamma}. \tag{40}$$

In this case, the correlation of the impurities via the chain does exist and will be seen in the Green functions of the system (Andrei and Johannesson 1984, Zhou and Jiang 1989). The resolution of this problem relies heavily on the solution of the Bethe ansatz equations and will be considered elsewhere.

Acknowledgments

We are thankful to Dr P F Wu and Mr Z Xiong for many discussions.

References

Andrei N, Furuya K and Lowenstein J H 1983 *Rev. Mod. Phys.* **5** 331
Andrei N and Johannesson H 1984 *Phys. Lett.* **100A** 108
Belavin A A 1979 *Phys. Lett.* **87B** 117
Bethe H 1931 *Z. Phys.* **79** 205
de Vega H J and Lopes E 1987 *Phys. Lett.* **186B** 180
Faddeev L D 1980 *Sov. Sci. Rev. Math. Phys.* C **1** 107
Faddeev L D and Reshetikhin N Yu 1985 *Ann. Phys., NY* **167** 215
Izergin A G and Korepin V E 1982 *Sov. J. Part. Nucl.* **13** 207
Kirillov A N and Reshetikhin N Yu 1987 *J. Phys. A: Math. Gen.* **20** 1565
Kulish P P and Sklyanin E K 1982 *Lecture Notes in Physics* vol 151, ed J Hierterinta and C Montonen (Berlin: Springer)
Lieb E and Liniger W 1963 *Phys. Rev.* **130** 1605
Thacker H B 1981 *Rev. Mod. Phys.* **53** 255
Wiegmann P B 1980 *Phys. Lett.* **80A** 163
Yang C N 1967 *Phys. Rev. Lett.* **19** 312
Zhou H Q and Jiang L J 1989 *Phys. Lett.* **134A** 469
Zhou H Q, Jiang L J and Wu P F 1989 *Phys. Lett.* **137A** 244